Positive stationary solutions for p-Laplacian problems with nonpositive perturbation

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Abstract

The paper is devoted to the existence of positive solutions of nonlinear elliptic equations with p-Laplacian. We provide a general topological degree that detects solutions of the problem

$$\begin{cases} A(u) = F(u) \\ u \in M \end{cases}$$

where $A : X \supset D(A) \to X^*$ is a maximal monotone operator in a Banach space X and $F : M \to X^*$ is a continuous mapping defined on a closed convex cone $M \subset X$. Next, we apply this general framework to a class of partial differential equations with p-Laplacian under Dirichlet boundary conditions. In the paper we employ general ideas from [5], where a setting suitable for the one dimensional p-Laplacian was introduced.

1 Introduction

We shall be concerned with solutions to the following nonlinear boundary value problem

$$(1) \qquad \begin{cases} -\mathrm{div}(|\nabla u(x)|^{p-2}\nabla u(x)) = f(x, u(x)), \ x \in \Omega, \\ u(x) \geq 0, \ x \in \Omega, \\ u(x) = 0, \ x \in \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) is a bounded domain with the smooth boundary $\partial\Omega$, $p \geq 2$ and $f : \Omega \times [0, +\infty) \to \mathbb{R}$ is a Carathéodory function ([4]). The differential term $\mathrm{div}(|\nabla u(x)|^{p-2}\nabla u(x))$ is referred to as the p-Laplacian of u at a point $x \in \Omega$. We search for *weak solutions* in the Sobolev space $W_0^{1,p}(\Omega)$, i.e. $u \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla v(x)\,\mathrm{d}x = \int_\Omega f(x, u(x))v(x)\,\mathrm{d}x \quad \text{for all} \ \ v \in W_0^{1,p}(\Omega).$$

Such boundary problems with p-Laplace were widely studied by many authors who used various methods. Let us mention just a few. Equations with the one dimensional p-Laplacian, i.e. when $N = 1$, were studied by Manásevich, Njoku i Zanolin [17], Drábek, García-Huidobro and Manásevich [7] and as well as by Kryszewski and the author [5]. In the general case, i.e. when $N > 1$, positive solutions of p-Laplace problems have been studied by a number of authors, e.g. Huang [13], Drábek and Pohozaev [6], Cañada, Drábek

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[4] Recall that we say that $f : \Omega \times [0, +\infty) \to \mathbb{R}$ is a Carathéodory function if $f(\cdot, s)$ is measurable for all $s \in [0, +\infty)$ and $f(x, \cdot)$ is continuous for almost all $x \in \Omega$.

and Gámez [3], Filippiakis, Gasiński and Papageorgiou [9] or Montreanu D., Montreanu V. V. and Papageorgiou [18], Väth [19].

Generally speaking, in the above mentioned papers, either $N = 1$ or N is arbitrary but the right has side of the equation - the function f is assumed to be non-negative or satisfy some monotonicity assumptions. This makes possible to apply Krasnosel'skii's fixed point theorem (in general, fixed point index in cones) or variational methods. These assumptions on f seem rather restrictive and sometimes unnatural, especially, when we take into account physical interpretation of the considered boundary value problem. In this paper, we do not require f to be non-negative or monotone. A general tool for detection of nonnegative solutions is provided. It is based on the geometric idea of tangency and using fixed point index in cones. We construct a topological degree for perturbations of maximal monotone operators with respect to closed convex cones. Next we prove appropriate index formulae, which together with the homotopy property, allow us to compute the topological degree in specific examples. It is noteworthy, that this setting does not require variational structure and can be also used for systems of p-Laplace problems. In this paper, we apply the method to show the following existence criterion

Theorem 1.1 *Suppose that a Carathéodory function $f : \Omega \times [0, +\infty) \to \mathbb{R}$ and $\rho_0, \rho_\infty \in L^\infty(\Omega)$ satisfy the following conditions*

(2) *there is $C > 0$ such that $|f(x,s)| \le C(1 + s^{p-1})$ for all $s \ge 0$ and a.a. $x \in \Omega$;*

(3) $\displaystyle\lim_{s\to 0^+} \frac{f(x,s)}{s^{p-1}} = \rho_0(x)$ *and* $\displaystyle\lim_{s\to\infty} \frac{f(x,s)}{s^{p-1}} = \rho_\infty(x)$ *uniformly with respect to $x \in \Omega$.*

If the principal eigenvalue $\lambda_{1,p}$ of the p-Laplace operator lies between ρ_0 and ρ_∞, i.e. either $\rho_0(x) < \lambda_{1,p} < \rho_\infty(x)$, for a.a. $x \in \Omega$, or $\rho_\infty(x) < \lambda_{1,p} < \rho_0(x)$, for a.a. $x \in \Omega$, then the problem (1) admits a nontrivial weak solution $u \in W_0^{1,p}(\Omega)$ such that $u(x) \ge 0$ for a.e. $x \in \Omega$.

Here the principal eigenvalue $\lambda_{1,p}$ is the smallest real number λ such that the problem

$$\begin{cases} -\operatorname{div}(|\nabla u(x)|^{p-2}\nabla u(x)) = \lambda |u(x)|^{p-2}u(x), \ x \in \Omega \\ u(x) \ge 0, \ x \in \Omega \\ u(x) = 0, \ x \in \partial\Omega \end{cases} \tag{4}$$

admits a nonzero weak solution (see Remark 4.5 for more details). Theorem 1.1 corresponds directly to the result of [13], obtained by different methods (the sub-supersolution technique and the existence result for variational inequalities) and under different assumptions corresponding to the inequality $\rho_\infty < \lambda_{1,p} < \rho_0$. Our general method allows us to consider also the case $\rho_\infty > \lambda_{1,p} > \rho_0$.

The paper is organized as follows. In Section 2 we develop a topological degree detecting coincidence points of maximal monotone operators and continuous operators in closed convex cones. This general tool will be useful if we rewrite the problem (1) in the form

$$\begin{cases} A_p u = N_f(u) \\ u \in M_p \end{cases}$$

where $A_p : L^p(\Omega) \supset D(A_p) \to L^p(\Omega)^*$ is the maximal monotone operator determined by the p-Laplacian, $N_f : L^p(\Omega) \to L^p(\Omega)^*$ is the Nemytzkii type operator associated with f and M_p is the closed convex cone of all non-negative elements in the space $L^p(\Omega)$. Section 3

provides a general setting in which assumptions of Section 2 are verified. Next, in Section 4 we show that the problem (1) falls into the setting and, using our topological degree together with spectral properties of p-Laplacian, we derive topological index formulae. They turn out to be essential in the proof of Theorem 1.1, which is provided at the end of Section 4.

Notation

If X is a metric space and $B \subset X$, then ∂B and clB stand for the boundary of B and the closure of B, respectively. If $x_0 \in X$ and $r > 0$, then $B(x_0, r) := \{x \in M \mid d(x, x_0) < r\}$.

If E is a normed space, then by $\|\cdot\|$ we denote its norm. If E is a normed space and E^* its dual space (of all continuous linear functionals), then $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_E : E^* \times E \to \mathbb{R}$ denotes the duality operator $\langle p, u\rangle := p(u)$, $p \in E^*$, $u \in E$. If V is another normed space then $\mathcal{L}(V, E)$ stands for the space of all bounded linear operators with domain V and values in E with the operator norm denoted by $\|\cdot\|_{\mathcal{L}(V,E)}$ or simply $\|\cdot\|$ if no confusion may appear.

For $x \in \mathbb{R}^N$, $N \geq 1$, $|x|$ denotes the Euclidean norm of x and $x \cdot y$ is the Euclidean scalar product of $x, y \in \mathbb{R}^N$.

2 Constrained topological degree for perturbations of maximal monotone operators

In this section we provide a construction of a topological degree detecting solutions of the abstract constrained problem

$$\begin{cases} 0 \in -Au + F(u) \\ u \in M \end{cases} \tag{5}$$

where $A : X \supset D(A) \multimap X^*$ is a densely defined maximal monotone operator, the constraint set M is a subset of X and $F : \overline{U} \to X^*$ is a continuous mapping defined on the closure of an open bounded $U \subset M$. Throughout the whole section we make the following assumptions

($\mathcal{A}_1$) there is a homeomorphism $N : X \to X^*$ such that N is bounded on bounded sets and the mappings $J_\alpha : X^* \to X$, $\alpha > 0$,

$$J_\alpha(\tau) := u, \quad \text{where } u \in D(A) \text{ is the unique element such that } \tau \in (N + \alpha A)(u),$$

are well defined and continuous;

($\mathcal{A}_2$) the mapping $\mathcal{J} : X^* \times (0, +\infty) \ni (\tau, \alpha) \mapsto J_\alpha(\tau) \in X$ is bounded on bounded sets and such that $\mathcal{J}_{\,|X^* \times [\alpha_1, \alpha_2]}$ is completely continuous if $0 < \alpha_1 \leq \alpha_2$;

($\mathcal{A}_3$) $M \subset X$ is a neighborhood retract of X, $J_\alpha(N(M)) \subset M$ for $\alpha > 0$, and $M^* := N(M)$ is an $\mathcal{L}$-retract (see [2] and [5]), i.e. there exist a retraction $r : B(M^*, \eta) \to M^*$ with some $\eta > 0$ and a constant $L > 0$ such that

$$\|r(\tau) - \tau\| \leq L d_{M^*}(\tau) \quad \text{for all} \quad \tau \in B(M^*, \eta); \tag{6}$$

$(\mathcal{A}_4)$ F is continuous, bounded on bounded sets and satisfies the tangency condition

$$F(N^{-1}(\tau)) \in T_{M^*}(\tau), \quad \text{for } \tau \in N(\overline{U}), \tag{7}$$

where $T_{M^*}(\tau)$ is the Bouligand tangent cone to M^* at the point τ, i.e.

$$T_{M^*}(\tau) := \left\{ \theta \in X^* \mid \liminf_{\alpha \to 0^+} \frac{d_{M^*}(\tau + \alpha\theta)}{\alpha} = 0 \right\}.$$

Remark 2.1 Since maximal monotone operators have closed graphs, it can be shown that in order to verify the continuity of the mapping $\mathcal{J}_{\mid X^* \times [\alpha_1, \alpha_2]}$ from condition $(\mathcal{A}_2)$ it is sufficient to know that it maps bounded sets into relatively compact ones.

Our goal is to transform the problem (5) into a fixed point one in M and for which fixed point index theory can be used. To this end define $\Phi_\alpha = \Phi_\alpha^{A,F} : \overline{U} \to M$ by

$$\Phi_\alpha(u) := J_\alpha \left(r \left(N(u) + \alpha F(u) \right) \right), \quad u \in \overline{U},$$

whenever $0 < \alpha < \eta / \sup\{\|F(u)\| \mid u \in \overline{U}\}$. Obviously, it is well defined, since for such α one has $(N + \alpha F)(\overline{U}) \subset B(M^*, \eta)$. Moreover, observe that due to the assumptions, the mapping $r \circ (N + \alpha F)$ is bounded on bounded sets and, by $(\mathcal{A}_2)$, Φ_α is compact.

Exploiting the tangency condition (7) and the inequality (6) together with compactness, we obtain the following localization of fixed points results.

Proposition 2.2 *If $K \subset \overline{U}$ is a closed set such that*

$$\{u \in \overline{U} \cap D(A) \mid 0 \in -Au + F(u)\} \cap K = \emptyset,$$

then, for sufficiently small $\alpha > 0$, $\{u \in \overline{U} \mid \Phi_\alpha(u) = u\} \cap K = \emptyset$.

Remark 2.3 Actually the tangency condition (7) and the continuity of $F \circ N^{-1}$ imply

$$F\left(N^{-1}(\tau)\right) \in C_{M^*}(\tau) := \left\{ \theta \in X^* \mid \lim_{\alpha \to 0^+,\, \varrho \to \tau,\, \varrho \in M} \frac{d_M(\varrho + \alpha\theta)}{\alpha} = 0 \right\} \ \text{ for all } \tau \in N(U).$$

Indeed

$$F\left(N^{-1}(\tau)\right) = \lim_{\varrho \to \tau} F\left(N^{-1}(\varrho)\right) \in \operatorname*{Liminf}_{\varrho \to \tau,\, \varrho \in M^*} T_{M^*}(\varrho) \subset C_{M^*}(\tau).$$

The proof of the latter inclusion can be found in [1].

Lemma 2.4 (i) *The graph* $\mathrm{Gr}(A) := \{(u, \tau) \in X \times X^* \mid u \in D(A)\}$ *is closed;*
(ii) *If a sequence of pairs $(u_n, \tau_n) \in \mathrm{Gr}(A)$, $n \geq 1$, is bounded, then the sequence (u_n) has a convergent subsequence.*

Proof: (i) Take any sequence of points $(u_n, \tau_n) \in \mathrm{Gr}(A)$, $n \geq 1$, such that $(u_n, \tau_n) \to (u_0, \tau_0)$ in $X \times X^*$, as $n \to +\infty$, for some $(u_0, \tau_0) \in X \times X^*$. Clearly, $\tau_n \in Au_n$, and this gives $N(u_n) + \tau_n \in (N + A)(u_n)$, which, by $(\mathcal{A}_1)$, gives $u_n = J_1(N(u_n) + \tau_n)$, $n \geq 1$. Hence, using the continuity of N and J_1 yields $u_n = J_1(N(u_n) + \tau_n)) \to J_1(N(u_0) + \tau_0)$

as $n \to +\infty$, which implies $u_0 = J_1(N(u_0) + \tau_0)$, i.e. $\tau_0 \in Au_0$. This shows that $\mathrm{Gr}(A)$ is closed.

(ii) Note that, for each $n \geq 1$, $u_n = J_1(N(u_n) + \tau_n) \in J_1(N(B(0, R)) + B(0, R))$, where $R > 0$ is a constant such that $\|u_n\|_X \leq R$ and $\|\tau_n\|_{X^*} \leq R$ for $n \geq 1$. The boundedness of N and $(\mathcal{A}_2)$ imply that the set (u_n) is a sequence of elements of the relatively compact set $J_1(N(B(0, R)) + B(0, R))$. □

Proof of Proposition 2.2: Suppose to the contrary that there exists a sequence (α_n) such that $\alpha_n \to 0^+$ such that for each $n \geq 1$ there is $u_n \in K$ with $\Phi_{\alpha_n}(u_n) = u_n$, that is

$$N(u_n) + \alpha_n \tau_n = r(N(u_n) + \alpha_n F(u_n)) \text{ for some } \tau_n \in Au_n.$$

In view of (6), one has

$$\begin{aligned} \alpha_n \|\tau_n - F(u_n)\| &= \|r(N(u_n) + \alpha_n F(u_n)) - (N(u_n) + \alpha_n F(u_n))\| \\ &\leq L d_{M^*}(N(u_n) + \alpha_n F(u_n)) \quad \text{for all } n \geq 1. \end{aligned} \tag{8}$$

This implies

$$\|\tau_n\| \leq \|F(u_n)\| + L\alpha_n^{-1} d_{M^*}(N(u_n) + \alpha_n F(u_n)) \leq (1 + L)\|F(u_n)\|, \ n \geq 1,$$

which means that (τ_n) is bounded. Therefore, by use of Lemma 2.4 (ii), we may assume without loss of generality that $u_n \to u_0$ for some $u_0 \in M$. Now using (8) and putting $p_n := N(u_n)$, $n \geq 0$, we see that

$$\|\tau_n - F(u_n)\| \leq L \cdot \frac{d_{M^*}(p_n + \alpha_n F(N^{-1}(p_0))}{\alpha_n} + L\|F(u_n) - F(u_0)\|, \quad \text{for } n \geq 1.$$

By the tangency condition $(\mathcal{A}_4)$ and Remark 2.3 together with the continuity of F, we get that $\tau_n \to F(u_n)$ as $n \to +\infty$. Hence, we have obtained that $(u_n, \tau_n) \to (u_0, F(u_0))$ and, by Lemma 2.4 (i), $(u_0, F(u_0)) \in \mathrm{Gr}(A)$, i.e. $F(u_0) \in Au_0$, a contradiction completing the proof. □

Now we put

$$\mathrm{Deg}_M(A, F, U) := \lim_{\alpha \to 0^+} \mathrm{ind}_M(\Phi_\alpha, U), \tag{9}$$

where ind_M stands for the fixed point index for compact mappings of absolute neighborhood retracts due to Granas – see [12] or [8] for details. We call this number as *the topological degree of coincidence (*or just *topological degree) of A and F with respect to M*.

Theorem 2.5 *The coincidence degree defined by* (9) *is well defined and has the following properties:*

(i) (existence) *if* $\mathrm{Deg}_M(A, F, U) \neq 0$*, then there exists* $u \in U \cap D(A)$ *such that* $0 \in -Au + F(u)$*;*

(ii) (additivity) *if* U_1, U_2 *are open disjoint subsets of a bounded open* $U \subset M$ *and* $0 \notin (-A + F)(\overline{U} \setminus (U_1 \cup U_2))$*, then*

$$\mathrm{Deg}_M(A, F, U) = \mathrm{Deg}_M(A, F, U_1) + \mathrm{Deg}_M(A, F, U_2);$$

(iii) (homotopy invariance) *if* $H:\overline{U}\times[0,1]\to X^*$ *is a continuous and bounded mapping such that*

$$H(N^{-1}(\tau),t)\in T_{M^*}(\tau) \text{ for all } \tau\in N(\overline{U}),\ t>0,$$

and $0\notin -Au+H(u,t)$ *for all* $u\in\partial U\cap D(A)$ *and* $t\in[0,1]$*, then*

$$\mathrm{Deg}_M(A,H(0,\cdot),U)=\mathrm{Deg}_M(A,H(1,\cdot),U);$$

(iv) (normalization) *if* M *is bounded and the mapping* $\widetilde{\mathcal{J}}: X^*\times[0,+\infty)\ni(\tau,\alpha)\mapsto J^\alpha\tau\in X$ *with* $J^0=N^{-1}$ *is continuous, then* $\mathrm{Deg}_M(A,F,M)=\chi(M)$.

Proof: Note that for sufficiently small $\alpha>0$ it follows from Propostion 2.2 that Φ_α has no fixed point in ∂U, i.e. the fixed point index $\mathrm{ind}_M(\Phi_\alpha,U)$ is well defined. If $\alpha_1,\alpha_2>0$ are small enough, then, by $(\mathcal{A}_2)$, Φ_{α_1} is homotopic with Φ_{α_2}, which gives $\mathrm{ind}_M(\Phi_{\alpha_1},U)=\mathrm{ind}_M(\Phi_{\alpha_2},U)$, which means that the limit in (9) exists.

(i) Suppose to the contrary that there is no $u\in U\cap D(A)$ such that $0\in -Au+F(u)$. Then, in view of Proposition 2.2, for sufficiently small $\alpha>0$ the mappings Φ_α have no fixed points in $\overline{U}$, i.e. $\mathrm{Deg}_M(A,F,U)=\mathrm{ind}_M(\Phi_\alpha,U)=0$, a contradiction.

(ii) Due to Proposition 2.2, for sufficiently small $\alpha>0$, Φ_α has no fixed points in $\overline{U}\setminus(U_1\cup U_2)$. Therefore, by the definition of the degree,

$$\mathrm{Deg}_M(A,F,U)=\mathrm{ind}_M(\Phi_\alpha,U)\quad\text{and}\quad\mathrm{Deg}_M(A,F,U_k)=\mathrm{ind}_M(\Phi_\alpha,U_k)\text{ for }k=1,2.$$

By the additivity property of the fixed point index

$$\mathrm{ind}_M(\Phi_\alpha,U)=\mathrm{ind}_M(\Phi_\alpha,U_1)+\mathrm{ind}_M(\Phi_\alpha,U_2),$$

which together with the earlier equalities gives the desired additivity of the degree.

(iii) For sufficiently small $\alpha>0$ one can define $\Phi_\alpha:\overline{U}\times[0,1]\to M$ by

$$\Phi_\alpha(u,t):=J_\alpha\left(r(N(u)+\alpha H(u,t))\right),\ u\in\overline{U},\ t\in[0,1].$$

Proceeding along the lines of the proof of Proposition 2.2 we can prove that for sufficiently small $\alpha>0$

$$\Phi_\alpha(u,t)\neq u \text{ for all } u\in\partial U,\ t\in[0,1].$$

Hence, by the homotopy invariance of the fixed point index and the formula defining the degree,

$$\mathrm{Deg}_M(A,H(\cdot,0),U)=\mathrm{ind}_M(\Phi_\alpha(\cdot,0),U)=\mathrm{ind}_M(\Phi_\alpha(\cdot,1),U)=\mathrm{Deg}_M(A,H(\cdot,1),U).$$

(iv) Take small $\alpha>0$ such that Φ_α is well defined. Then

$$\mathrm{Deg}_M(A,F,M)=\mathrm{ind}_M(\Phi_\alpha,M).$$

Note that the normalization property for the fixed point index states that the homomorphism $H_*(\Phi_\alpha): H_*(M)\to H_*(M)$ induced on (singular) homology spaces is a Leray endomorphism and

$$\mathrm{ind}_M(\Phi_\alpha,M)=\Lambda(\Phi_\alpha) \tag{10}$$

where $\Lambda(\Phi_\alpha)$ is the generalized Leschetz number of the compact map Φ_α – see [8, Definition V.(2.1), (3.1) and Theorem (5.1)] or [12]. Further, consider $\Psi: M\times[0,1]\to M$ given by

$$\Psi(u,t):=\widetilde{\mathcal{J}}(r(N(u)+t\alpha F(u)),t\alpha),\quad u\in M,\quad t\in[0,1].$$

By the assumption, Ψ is a continuous homotopy joining $\Psi(\cdot, 1) = \Phi_\alpha$ with the identity map $\mathrm{id}_M : M \to M$. Hence, for the maps induced on homology spaced one has $H_*(\Phi_\alpha) = H_*(\mathrm{id}_M) = \mathrm{id}_{H_*(M)}$ and, since $H_*(\Phi_\alpha)$ is an endomorphism Leray, we infer that $\Lambda(\Phi_\alpha) = \sum_{n=0}^{\infty}(-1)^n \dim H_n(M) = \chi(M)$, which together with (10) ends the proof. $\square$

We end this section with a general result, which allows us to compute the degree is specific situations (comp. [5, Prop. 4.2]).

Theorem 2.6 *Let M and M^* be closed convex cones and that the mappings A and N are homogeneous with the same degree* ([5])*. Suppose that there exists $\lambda_1 \geq 0$ satisfying the following conditions*

$(\mathcal{M}_1)$ $(A - \lambda N)^{-1}(\{0\}) \cap M = \{0\}$ *for* $\lambda \neq \lambda_1$;

$(\mathcal{M}_2)$ *there exists $\tau_0 \in M^*$ such that $(A - \lambda N)^{-1}(\{\tau_0\}) \cap M = \emptyset$ for $\lambda > \lambda_1$.*

Then

$$\mathrm{Deg}_M(A, \lambda N, B_M(0, \delta)) = \begin{cases} 1, & \lambda < \lambda_1, \\ 0, & \lambda > \lambda_1, \end{cases}$$

for any $\delta > 0$.

Proof: Note that in view of $(\mathcal{M}_1)$ the topological degree $\mathrm{Deg}_M(A, \lambda N, B_M(0, \delta))$ is well defined.

Now fix $\lambda < \lambda_1$. By the very construction, for sufficiently small $\alpha > 0$,

$$\mathrm{Deg}_M(A, \lambda N, B_M(0, \delta)) = \mathrm{ind}_M(\Phi_\alpha, B_M(0, \delta)) \tag{11}$$

where $\Phi_\alpha : \overline{B_M(0, \delta)} \to M$ is given by

$$\Phi_\alpha(u) := J_\alpha(r(N(u) + \alpha\lambda N(u))),\ u \in \overline{B_M(0, \delta)}.$$

Define $\Theta : \overline{B_M(0, \delta)} \times [0, 1] \to M$ by

$$\Theta(u, t) := t\Phi_\alpha(u),\ u \in \overline{B_M(0, \delta)},\ t \in [0, 1].$$

Suppose there are $u \neq 0$ and $t \in [0, 1]$ such that $\Theta(u, t) = u$. Then $0 \in -A(u) + \mu N(u)$ with $\mu := (t^\gamma - 1)/\alpha + t^\gamma\lambda$, i.e. $u \in (A - \mu N)^{-1}(\{0\}) \cap M$, and, since $\mu = (t^\gamma - 1)/\alpha + t^\gamma\lambda < \lambda_1$ we get a contradiction with $(\mathcal{M}_1)$. Hence, we can use the homotopy invariance of fixed point index to see that $\mathrm{ind}_M(\Phi_\alpha, B_M(0, \delta)) = \mathrm{ind}_M(0, B_M(0, \delta)) = 1$. This along with (11) implies the required equality.

Let us pass to the case when $\lambda > \lambda_1$. Define $H : M \times [0, 1] \to X$ by $H(u, t) := \lambda N(u) + t\tau_0$, $u \in M$, $t \in [0, 1]$. If $-A(u) + H(u, t) = 0$, then either $t = 0$ and, due to $(\mathcal{M}_1)$, $u = 0$ or, by the homogeneity $-A(t^{-1/\gamma}u) + \lambda N(t^{-1/\gamma}u) + \tau_0 = 0$, where $\gamma > 0$ is the common homogeneity degree for A and N. The latter equality contradicts $(\mathcal{M}_2)$. Hence, the degrees $\mathrm{Deg}_M(A, H(\cdot, t), B_M(0, \delta))$, $t \in [0, 1]$, are well defined and homotopy invariance can be used to obtain

$$\mathrm{Deg}_M(A, \lambda N, B_M(0, \delta)) = \mathrm{Deg}_M(A, \lambda A + \tau_0, B_M(0, \delta).$$

Finally the existence property of the degree together with $(\mathcal{M}_2)$ implies $\mathrm{Deg}_M(A, \lambda A + \tau_0, B_M(0, \delta)) = 0$, which completes the proof. $\square$

[5] i.e. there exists $\gamma > 0$ such that $A(au) = a^\gamma A(u)$, $u \in D(A)$, $a > 0$, and $N(au) = a^\gamma N(u)$ for all $u \in X$, $a > 0$.

3 Abstract setting for p-Laplacian

Now we shall consider an abstract example falling into the setting of Section 2. It will be used in the sequel for the p-Laplace operator and the cone of positive functions in $L^p(\Omega)$.

Let X and Y be reflexive normed spaces with a dense and compact linear embedding $i : Y \to X$.([6]) Suppose that a closed convex cone $M \subset X$ and functionals $\mathbf{a} : Y \to \mathbb{R}$ and $\mathbf{n} : X \to \mathbb{R}$ satisfy the following conditions:

(a1) $\mathbf{a}$ and $\mathbf{n}$ are coercive C^1 functionals; ([7])

(a2) there exists a continuous function $\kappa : [0, +\infty) \to [0, +\infty)$ such that $\kappa^{-1}(\{0\}) = \{0\}$, $\lim_{s\to+\infty} \kappa(s) = +\infty$ and

$$\langle D\mathbf{a}(u_1) - D\mathbf{a}(u_2), u_1 - u_2\rangle_Y \geq \kappa(\|u_1 - u_2\|_Y)\|u_1 - u_2\|_Y \quad \text{for all } u_1, u_2 \in Y,$$
$$\langle D\mathbf{n}(u_1) - D\mathbf{n}(u_2), u_1 - u_2\rangle_X \geq \kappa(\|u_1 - u_2\|_X)\|u_1 - u_2\|_X \quad \text{for all } u_1, u_2 \in X;$$

(a3) for any $u \in M$ there exist $u^+, u^- \in M$ such that $u = u^+ - u^-$ and $\mathbf{n}(u^+) \leq \mathbf{n}(u)$; if $u \in i(Y)$, then $u^+, u^- \in i(Y)$ and $\mathbf{a}(i^{-1}u^+) \leq \mathbf{a}(i^{-1}u)$;

(a4) $\mathbf{n}$ is bounded on bounded sets and monotone with respect to M, i.e. $\mathbf{n}(u+v) \geq \mathbf{n}(u)$ for any $u, v \in M$.

Let $\mathcal{A} : Y \to Y^*$ and $N : X \to X^*$ be defined by by $\mathcal{A} := D\mathbf{a}$ and $N := D\mathbf{n}$. Note that that, due to (**a2**), both $\mathbf{a}$ and $\mathbf{n}$ are strictly convex and $\mathcal{A}$ and N are monotone operators. Define $A : D(A) \to X^*$ by

$$D(A) := i\left(\mathcal{A}^{-1}(i^*(X^*))\right) \text{ and } Au := (i^*)^{-1}(\mathcal{A}i^{-1}u), \text{ for } u \in D(A). \tag{12}$$

The above operation of restriction is a generalization of the analogical one that is usually considered in the case of a Gelfand triple $Y \subset X \subset Y^*$ where X is a Hilbert space.

Below we show that assumptions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ of Section 2 are satisfied.

Proposition 3.1 *Under the above assumptions*
(i) *N is a homeomorphism which is bounded on bounded sets;*

(ii) $N(M) = M^* := \{\tau \in X^* \mid \langle \tau, u\rangle \geq 0$ *for all* $u \in M\}$*;*

(iii) *A is a densely defined maximal monotone operator;*

(iv) *for any $\alpha > 0$ and $\tau \in X^*$ there is a unique $u \in D(A)$ such that $\tau = (N + \alpha A)(u)$;*

[6] That is the mapping i is linear and completely continuous with its range $i(Y)$ dense in X.
[7] By *coercivity* we mean that counterimages of intervals $(-\infty, m)$, with respect to a given functional, are bounded for all $m \in \mathbb{R}$.

(v) *if* $J_\alpha : X^* \to X$, $\alpha > 0$, *is given by*

$$J_\alpha \tau := u \text{ where } u \in D(A) \text{ is such that } N(u) + \alpha A(u) = \tau,$$

and $\mathcal{J} : X^* \times [0, +\infty) \to X$ *by*

$$\mathcal{J}(u, \alpha) := J_\alpha u,$$

then $\mathcal{J}$ *is bounded on bounded sets and* $\mathcal{J}_{\,|X^* \times [\alpha_1, \alpha_2]}$ *with* $0 < \alpha_1 \le \alpha_2$ *is completely continuous;*

(vi) $J_\alpha(M^*) \subset M$ *for all* $\alpha > 0$.

Proof: To see (i), first note that N is continuous, since $\mathbf{n}$ is C^1. Moreover, as a strictly convex coercive functional on the reflexive Banach space X, for any $\tau \in X^*$, $\mathbf{n} - \tau$ admits a unique minimum point $u \in X$, i.e. $D\mathbf{n}(u) - \tau = 0$, which gives $N(u) = \tau$. Conversely, if $u \in X$ is such that $N(u) = \tau$, then, by the strict convexity, u is the unique minimum point. Hence, N is bijective. To see that N^{-1} is continuous, take any (τ_n) in X with $\tau_n \to \tau$ in X^* as $n \to +\infty$. Observe that, by (**a2**), we get

$$\langle \tau_n - \tau, N^{-1}(\tau_n) - N^{-1}(\tau)\rangle_X \ge \kappa(\|N^{-1}(\tau_n) - N^{-1}(\tau)\|_X)\|N^{-1}(\tau_n) - N^{-1}(\tau)\|_X,$$

which yields the inequality

$$\|\tau_n - \tau\|_{X^*} \ge \kappa\left(\|N^{-1}(\tau_n) - N^{-1}(\tau)\|_X\right).$$

This in turn means that $N^{-1}(\tau_n) \to N^{-1}(\tau)$ in X as $n \to +\infty$, that is N^{-1} is continuous.

To show that N is bounded on bounded sets, we suppose to the contrary that there exists a bounded sequence (u_n) in X such that $\|N(u_n)\|_{X^*} \to +\infty$ as $n \to +\infty$. Since X is reflexive, for each $n \ge 1$ one finds an element $v_n \in X$ such that $\|v_n - u_n\|_X = 1$ and

$$\|N(u_n)\|_{X^*} = \langle N(u_n), v_n - u_n\rangle \le \mathbf{n}(v_n) - \mathbf{n}(u_n) \le \sup_{D_X(0,R+1)} \mathbf{n} - \inf_{D_X(0,R)} \mathbf{n}$$

where $R > 0$ is such that $\|u_n\| \le R$ for all $n \ge 1$. Thus, a contradiction proving the claim.

To get (ii) take any $u \in M$ and $v \in M$. In view of (**a4**)

$$\mathbf{n}(u + hv) - \mathbf{n}(u) \ge 0 \quad \text{for any } h > 0,$$

which, after a division by h and passage to the limit with $h \to 0^+$, yields $\langle N(u), v\rangle \ge 0$. Hence $N(M) \subset M^*$. To prove the converse inclusion $M^* \subset N(M)$, we take any $\tau \in M^*$. As we mentioned $\mathbf{n} - \tau$ attains the minimum at some $u \in X$. On the other hand, by (**a2**),

$$\mathbf{n}(u^+) - \tau(u^+) \le \mathbf{n}(u) - \tau(u^+) + \tau(u^-) = \mathbf{n}(u) - \tau(u).$$

and, since the minimum point is unique, we infer that $u = u^+ \in M$.

To show (iii), take any $u_1, u_2 \in D(A)$. Clearly $(Au_k) \circ i = \mathcal{A}(\tilde{u}_k)$ with $\tilde{u}_k = i^{-1}(u_k)$, for $k = 1, 2$. Therefore, by (**a2**),

$$\langle Au_1 - Au_2, u_1 - u_2\rangle_X = [Au_1 - Au_2] i(\tilde{u}_1 - \tilde{u}_2) = \langle \mathcal{A}(\tilde{u}_1) - \mathcal{A}(\tilde{u}_2), \tilde{u}_1 - \tilde{u}_2\rangle_Y \ge 0.$$

Hence A is monotone and it is left to prove that A is maximal monotone, i.e. that additionally one has $A(D(A)) = X^*$. To see it we choose any $\tau \in X^*$ and put $\Phi := \mathbf{a} - i^*(\tau)$. Φ is a convex coercive functional on the reflexive space Y. Hence it admits a miniumem, i.e. there is a point $\bar{u} \in Y$ such that $D\Phi(\bar{u}) = 0$, i.e. $\mathcal{A}(\bar{u}) = i^*(\tau)$. This means that

$u := i(\bar{u}) \in D(A)$ and that $A(u) = \tau$.

To show (iv) take any $\tau \in X^*$ and $\alpha > 0$. We proceed like in (iii), that is we consider a functional $\Phi := \mathbf{n} \circ i + \alpha \mathbf{a} - i^*(\tau)$ on Y. It is clear that Φ – as a strictly convex and coercive functional on a reflexive Banach space – admits a minimum, i.e. there exists $\bar{u} \in Y$ such that $D\Phi(\bar{u}) = 0$. This means that $i^*(N(i(\bar{u}))) + \alpha \mathcal{A}(\bar{u}) = i^*(\tau)$. Subsequently, we deduce that $\mathcal{A}(\bar{u}) \in i^*(X^*)$, i.e. $u := i(\bar{u}) \in D(A)$ and $N(u) + \alpha A(u) = \tau$. Moreover, observe that for each $u \in D(A)$ such that $N(u) + \alpha A(u) = \tau$, $i^{-1}(u)$ is a critical point of Φ. Since Φ is strictly convex it has to be the unique minimum point.

(v) Suppose that a sequence (τ_n) is bounded in X^* and (β_n) is a sequence in $[\alpha_1, \alpha_2]$. Put $u_n := J_{\beta_n}(\tau_n)$, $n \geq 1$. Then $i^* N(u_n) + \beta_n \mathcal{A}(\bar{u}_n) = i^*(\tau_n)$, where $\bar{u}_n := i^{-1}(u_n)$, $n \geq 1$. Since N is bounded and $\beta_n > \alpha_1 > 0$ for all $n \geq 1$, we infer that $(\mathcal{A}(\bar{u}_n))$ is bounded. Observe that, in view of (**a2**),

$$\langle \mathcal{A}(\bar{u}_n) - \mathcal{A}(0), \bar{u}_n \rangle_Y \geq \kappa(\|\bar{u}_n\|_Y) \|\bar{u}_n\|_Y,$$

i.e. $\|\mathcal{A}(u_n) - \mathcal{A}(0)\|_Y \geq \kappa(\|\bar{u}_n\|_Y)$. Hence, by the boundedness of $(\mathcal{A}(\bar{u}_n))$ and the assumed property of κ, $(\bar{u}_n)$ is bounded. Therefore $(u_n) = (i(\bar{u}_n))$ is relatively compact, which together with Remark 2.1 proves the assertion.

In order to prove (vi), take any $\tau \in M^*$. We need to show that $u := J_\alpha(\tau) \in M$. In the proof of (iv) we have showed that $i^{-1}u$ is the unique minimum of the functional $\Phi = \mathbf{n} \circ i + \alpha \mathbf{a} - i^*(\tau)$ on Y. On the other hand, by use of (**a3**) and the definition of M^*, one has

$$\Phi(i^{-1}u^+) = \mathbf{n}(u^+) + \alpha \mathbf{a}(i^{-1}u^+) - \tau(u^+) \leq \mathbf{n}(u) + \alpha \mathbf{a}(i^{-1}u)) - \tau(u^+) + \tau(u^-) = \Phi(i^{-1}u).$$

This means that $i^{-1}u = i^{-1}u^+$ and $u \in M$. $\square$

4 Elliptic problems with p-Laplacian

Now we shall apply the above abstract setting from the previous section to the p-Laplacian problem. To this end fix $p > 2$, and put

$$X_p := L^p(\Omega),\ Y_p := W_0^{1,p}(\Omega) \text{ and } M_p := \{u \in X \mid u(x) \geq 0 \text{ for a.e. } x \in \Omega\}.$$

Both, X_p and Y_p are reflexive and, by the Rellich-Kondrachov theorem, the natural embedding $i : Y_p \to X_p$ is compact and dense. It is easy to see that M_p is a closed convex subset of X_p. Next define functionals $\mathbf{a} : Y_p \to \mathbb{R}$ and $\mathbf{n} : X_p \to \mathbb{R}$ by

$$\mathbf{a}(u) := \frac{1}{p} \int_\Omega |\nabla u(x)|^p \,\mathrm{d}x,\ u \in Y_p,$$
$$\mathbf{n}(u) := \frac{1}{p} \int_\Omega |u(x)|^p \,\mathrm{d}x,\ u \in X_p.$$

We prove that these objects satisfy the abstract assumptions of the general setting.

Proposition 4.1 *The functionals* $\mathbf{a}$ *and* $\mathbf{n}$ *with the cone* M_p *satisfy all the assumptions* **(a1)** – **(a4)** *from Section* 3 *and*

$$\langle D\mathbf{a}(u), v \rangle_Y = \frac{1}{p} \int_\Omega |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla v(x) \,\mathrm{d}x,\ u, v \in Y_p, \tag{13}$$

$$\langle D\mathbf{a}(u) - D\mathbf{a}(v), u - v\rangle_Y \geq 2^{2-p}\|u - v\|_Y^p, \; u, v \in Y_p, \tag{14}$$

$$\langle D\mathbf{n}(u), v\rangle_X = \frac{1}{p}\int_\Omega |u(x)|^{p-2}u(x)v(x)\,\mathrm{d}x, \;\; u, v \in X_p, \tag{15}$$

$$\langle D\mathbf{n}(u) - D\mathbf{n}(v), u - v\rangle_X \geq 2^{2-p}\|u - v\|_X^p, \; u, v \in X_p. \tag{16}$$

Moreover, if $A_p : D(A_p) \to X_p$ is defined, in analogy to (12), by

$$D(A_p) := i\left((D\mathbf{a})^{-1}(i^*(X^*))\right) \;\text{ and } A_p u := (i^*)^{-1}(D(\mathbf{a})i^{-1}u), \;\text{ for } u \in D(A),$$

then

$$A_p u = -\mathrm{div}(|\nabla u|^{p-2}\nabla u), \;\text{ for } u \in D(A_p),$$

where the divergence is meant in the distributional sense and

$$D(A_p) = \{u \in W_0^{1,p}(\Omega) \mid \mathrm{div}(|\nabla u|^{p-2}\nabla u) \text{ exists and belongs to } L^p(\Omega)\}.$$

Proof: In order to see **(a1)**, note that the functionals $\mathbf{a}$ and $\mathbf{n}$ are clearly Gateaux differentiable with the formulas (13) and (15) satisfied. Since these Gateaux derivatives are continuous the functionals are Frêchet differentiable. The coercivity is immediate as $\mathbf{a}(u) = (1/p)\|u\|_{Y_p}^p$, $u \in Y$, and $\mathbf{n}(u) = (1/p)\|u\|_{X_p}^p$, $u \in X$.
One can check the condition **(a2)**, i.e. (14) and (16), by use of the following inequality

$$(|x|^{p-2}x - |y|^{p-2}y)\cdot(x - y) \geq 2^{2-p}|x - y|^p \text{ for any } x, y \in \mathbb{R}^M, \; M \geq 1. \tag{17}$$

Obviously, for $\kappa : [0, +\infty) \to [0, +\infty)$, given by $\kappa(s) := 2^{2-p}s^p$, $s \geq 0$, one has $\kappa^{-1}(\{0\}) = \{0\}$, $\lim\limits_{s\to+\infty}\kappa(s) = +\infty$.

As for **(a3)**, take any $u \in X$. Then taking $u_+ := \max\{u, 0\}$ and $u_- := \max\{-u, 0\}$ we have $u = u^+ - u^-$ and

$$\mathbf{n}(u_+) = \frac{1}{p}\int_\Omega |u_+(x)|^p\,\mathrm{d}x \leq \frac{1}{p}\int_\Omega |u(x)|^p\,\mathrm{d}x = \mathbf{n}(u).$$

If $u \in Y_p = W_0^{1,p}(\Omega)$, then, due to Lemma 7.6 of [11], $\nabla u_+(x) = 0$ if $u(x) \leq 0$ and $\nabla u_+(x) = \nabla u(x)$ if $u(x) \geq 0$. Therefore $u_+ \in Y_p$ and

$$\mathbf{a}(u_+) = \frac{1}{p}\int_\Omega |\nabla u_+(x)|^p\,\mathrm{d}x \leq \frac{1}{p}\int_\Omega |\nabla u(x)|^p\,\mathrm{d}x = \mathbf{a}(u).$$

Finally, **(a4)** is immediate as, for $u, v \in M_p$, $|u|^p = u^p \leq (u + v)^p = |u + v|^p$. □

In view of Section 3, the operators A_p, $N_p := D\mathbf{n}$ together with M_p and M_p^* satisfy the assumptions made in Section 2 and the topological degree can be applied for perturbations of A_p. Before we proceed further let us pay attention to the perturbation term.

Proposition 4.2 *Let $f : \Omega \times [0, +\infty) \to \mathbb{R}$ satisfy (2) and $f(x, 0) \geq 0$ for a.a. $x \in \Omega$. Then the mapping $F : X_p \to X_p^*$ given by*

$$\langle F(u), v\rangle_{X_p} := \int_\Omega f(u(x))v(x)\,\mathrm{d}x, \quad u \in M_p, \, v \in X_p,$$

is well defined, continuous, bounded on bounded sets and

$$F(N^{-1}(\tau)) \in T_{M_p^*}(\tau) \quad \text{for any } \; \tau \in M_p^*. \tag{18}$$

Lemma 4.3 *Let* $1 < q < \infty$, $\Omega \subset \mathbb{R}^N$, $N \geq 1$, *be open and*

$$M_q := \{u \in L^q(\Omega) \mid u(x) \geq 0 \text{ for a.e. } x \in \Omega\}.$$

Then $\quad T_{M_q}(u) = \{v \in L^q(\Omega) \mid v(x) \geq 0 \text{ for a.e. } x \in \Omega \text{ such that } u(x) = 0\}.$

Proof: Put $T_u := \{v \in L^q(\Omega) \mid v(x) \geq 0$ for a.e. $x \in \Omega$ such that $u(x) = 0\}$. To see that $T_u \subset T_{M_q}(u)$ take any $v \in T_u$ and define $v_n \in L^1(\Omega)$, $n \geq 1$, by

$$v_n(x) := \begin{cases} v(x) & \text{if } v(x) + nu(x) \geq 0, \\ 0 & \text{if } v(x) + nu(x) < 0. \end{cases}$$

Clearly, $v_n \in M_q - nu \subset T_{M_q}(u)$, for each $n \geq 1$. Moreover it is clear that, for a.e. $x \in \Omega$ and any $n \geq 1$, $v_n(x) = v(x) \geq 0$ if $u(x) = 0$ and $v_n(x) \to v(x)$ if $u(x) > 0$. This implies that $v_n \to v$ in $L^q(\Omega)$, i.e. $v \in T_{M_q}(u)$.

In order to show the converse inclusion, observe that T_u is closed and, for any $h > 0$, $h(M - u) \subset T_u$. This clearly implies that $T_{M_q}(u) \subset T_u$. □

Proof of Proposition 4.2: Using the Riesz representation isomorphism ϱ between $L^p(\Omega)^*$ and $L^q(\Omega)$, $1/p + 1/q = 1$, the mapping $F \circ N^{-1}$ can be treated as the mapping $L^q(\Omega) \ni u \mapsto f(\cdot, \theta_q(u)) \in L^q(\Omega)$ where $\theta_q : \mathbb{R} \to \mathbb{R}$ is given by $\theta_q(s) = |s|^{q-2}s$, $s \in \mathbb{R}$. It is well defined as

$$|(f(x, \theta_q(s))| \leq C(1 + |\theta_q(s)|^p) = C(1 + |s|) \text{ for } s \geq 0 \text{ and a.e. } x \in \Omega.$$

Observe that $f(x, \theta_q(0)) = f(x, 0) \geq 0$ for. a.e. $x \in \Omega$, which, by use of Lemma 4.3, implies that $f(\cdot, \theta_q(u(\cdot))) \in T_{M_q}(u)$ for all $u \in M_q$. Since $\varrho(M_p^*) = M_q$, we infer that (18) holds. □

Hence we have showed that the problem (1) indeed can be formulated as an abstract problem

$$\begin{cases} A_p(u) = F(u), \\ u \in M_p \cap D(A_p). \end{cases}$$

In order to take advantage of the topological degree effectively we need some methods of computing it.

Theorem 4.4 *If* $2 < p < \infty$ *and* $\rho \in L^\infty(\Omega)$ *is such that either* $\rho(x) > \lambda_{1,p}$ *for a.e.* $x \in \Omega$, *or* $\rho(x) < \lambda_{1,p}$ *for a.e.* $x \in \Omega$, *then*

$$\mathrm{Deg}_{M_p}(A_p, \rho N_p, B_{M_p}(0, R)) = \begin{cases} 1, & \text{if } \rho(x) < \lambda_{1,p} \text{ for a.e. } x \in \Omega, \\ 0, & \text{if } \rho(x) > \lambda_{1,p} \text{ for a.e. } x \in \Omega. \end{cases}$$

Remark 4.5 Before passing to the proof of Theorem 4.4, we need to make a comment on the eigenvalue problem relating to the p-Laplace operator. Solving the nonlinear eigenvalue problem

$$\begin{cases} A_p(u) = \lambda N_p(u) \\ u \in M_p \cap D(A_p) \end{cases}$$

reduces to find nonnegative weak solutions $u \in W^{1,p}(\Omega)$ of

$$\begin{cases} -\mathrm{div}(|\nabla u(x)|^{p-2} \nabla u(x)) = \lambda |u(x)|^{p-2} u(x), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{19}$$

It appears that some properties of the eigenvalue problem for the Laplace operator are also valid for the p-Laplace one. For details we refer to [14], [15] and [16]. In particular, it is known that (19) does not admit any nonzero solutions if $\lambda \leq 0$, i.e. the p-Laplace has no nonpositive eigenvalues. Moreover, there exists the smallest eigenvalue $\lambda_{1,p}$ given by the Rayleigh formula

$$\lambda_{1,p} = \inf_{u \in W_0^{1,p}(\Omega), u \neq 0} \frac{\int_\Omega |\nabla u(x)|^p \, \mathrm{d}x}{\int_\Omega |u(x)|^p \, \mathrm{d}x}.$$

The eigenfunctions corresponding to $\lambda_{1,p}$ are either strictly positive or negative in Ω and belong to $L^\infty(\Omega)$. Moreover, $\lambda_{1,p}$ is an isolated eigenvalue and if there are two eigenfunctions u, v for $\lambda_{1,p}$, then there exists $\alpha \in \mathbb{R}$ such that $u = \alpha v$. It is also known that if any eigenfunction does not change its sign in Ω, then the corresponding eigenvalue must be equal to $\lambda_{1,p}$. □

In the proof we shall use a few lemmata given below.

Lemma 4.6 *There are $C, s > 0$ such that $\|u\|_{L^p} \leq C|\tilde{\Omega}|^s \|\nabla u\|_{L^p}$ for all $u \in W_0^{1,p}(\Omega)$ and measurable $\tilde{\Omega} \subset \Omega$ with the property $u(x) = 0$ if $x \notin \tilde{\Omega}$.*

Proof: By the Sobolev embedding theorem there exists $q > p$ such that

$$\|u\|_{L^q} \leq C\|\nabla u\|_p \text{ for all } u \in W_0^{1,p}(\Omega).$$

On the other hand, by the Hölder inequality,

$$\|u\|_{L^p} \leq \|u\|_{L^q} |\tilde{\Omega}|^{1/p - 1/q}.$$

Combining the two above inequalities we get the desired one with $s := 1/p - 1/q$. □

Lemma 4.7 *Let v be a nonnegative weak solution of* (19) *with $\lambda = \lambda_{1,p}$ and $\rho \in L^\infty(\Omega)$. If $u \in W_0^{1,p}(\Omega)$ is a weak solution to*

$$-\mathrm{div}(|\nabla u|^{p-2} \nabla u) = \rho |u|^{p-2} u + |v|^{p-2} v \quad \textit{on } \Omega,$$

then $u \in L^\infty(\Omega)$.

Proof: Here we adapt the arguments from [15]. Note that without loss of generality we can consider the equation

$$-\mathrm{div}(|\nabla u|^{p-2} \nabla u) = \rho |u|^{p-2} u + \lambda_{1,p} |v|^{p-2} v \text{ on } \Omega.$$

Take any $k > 0$ and put $\eta := \max\{u - v - k, 0\}$. Since $\eta \in W_0^{1,p}(\Omega)$, we get

$$\int_{\Omega_k} (|\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v) \cdot \nabla (u - v) \, \mathrm{d}x \leq \|\rho\|_{L^\infty} \int_{\Omega_k} u^{p-1} (u - v - k) \, \mathrm{d}x$$

with $\Omega_k := \{x \in \Omega \mid u(x) - v(x) - k > 0\}$. This, by use of (17) and the convexity of the function $s \mapsto |s|^{p-1}$, gives

$$\begin{aligned} \int_{\Omega_k} |\nabla (u - v)|^p \, \mathrm{d}x &\leq C_1 \int_{\Omega_k} u^{p-1} (u - v - k) \, \mathrm{d}x \\ &\leq C_1 2^{p-2} \left(\int_{\Omega_k} (u - v - k)^p \, \mathrm{d}x + \int_{\Omega_k} (v + k)^{p-1} (u - v - k) \, \mathrm{d}x \right) \end{aligned}$$

for some constant $C_1 > 0$ (here all the constant are to be independent of k). By applying Lemma 4.6, one gets

$$\int_{\Omega_k} (u - v - k)^p \, \mathrm{d}x \leq C|\Omega_k|^s \int_{\Omega_k} |\nabla(u - v)|^p \, \mathrm{d}x,$$

which together with the previous inequality yields

$$(1 - C_2|\Omega_k|^s) \int_{\Omega_k} (u - v - k)^p \, \mathrm{d}x \leq C_2|\Omega_k|^s \int_{\Omega_k} (v + k)^{p-1}(u - v - k) \, \mathrm{d}x$$

for some $C_2 > 0$. Since $|\Omega_k| \to 0$ as $k \to +\infty$, there is k_0 such that for all $k \geq k_0$ $1 - C_2|\Omega_k|^s > 1/2$. Further, for $k \geq k_0$,

$$\int_{\Omega_k} (u - v - k)^p \, \mathrm{d}x \leq 2C_2|\Omega_k|^s(\|v\|_{L^\infty} + k)^{p-1} \int_{\Omega_k} (u - v - k) \, \mathrm{d}x.$$

Next we observe that the Hölder inequality yields

$$\int_{\Omega_k} (u - v - k) \, \mathrm{d}x \leq C_4 k|\Omega_k|^{1+s(p-1)^{-1}} \text{ for all } k \geq k_0 \tag{20}$$

and some constant $C_4 > 0$. Now define $j : (0, +\infty) \to [0, +\infty)$ by

$$j(k) := \int_{\Omega_k} (u - v - k) \, \mathrm{d}x, \;\; k > 0.$$

Note that by the Tonelli-Fubini theorem applied to the set $\{(x, t) \in \Omega \times [0, +\infty) \mid u(x) - v(x) > t > k\}$ one has

$$j(k) = \int_k^{+\infty} |\Omega_t| \, \mathrm{d}t, \;\; k > 0.$$

Obviously, j is nonincreasing and absolutely continuous with $j'(k) = -|\Omega_k|$ for a.e. $k \geq 0$. We claim that $j(k) = 0$ for some $k > 0$. If it were not so, then (20) could be rewritten as

$$j(k)^\theta \leq -C_4^\theta k^\theta j'(k) \text{ for all } k \geq k_0$$

with $\theta := (1 + s(p - 1)^{-1})^{-1}$, and consequently

$$k^{-\theta} \leq -C_4^\theta j(k)^{-\theta} j'(k) \text{ for all } k \geq k_0.$$

This after integration would give

$$k^{1-\theta} + C_4^\theta j(k)^{1-\theta} \leq k_0^{1-\theta} + C_4^\theta j(k_0)^{1-\theta} \text{ for all } k \geq k_0,$$

which yields a contradiction proving the claim that $j(k) = 0$ for some $k > 0$. Then, for some $k > 0$, $|\Omega_k| = 0$ and $u \leq v + k$ a.e. on Ω. This shows that $u \in L^\infty(\Omega)$, as $v \in L^\infty(\Omega)$ (see Remark 4.5). □

Lemma 4.8 (see [10, Th. 1]) *If $h \in L^\infty(\Omega)$ is nonnegative and nonzero, then the equation*

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \lambda_{1,p}|u|^{p-2}u + h, \;\; \text{on } \Omega,$$

has no nonzero weak solution in $W_0^{1,p}(\Omega)$.

Lemma 4.9 *If $\rho \in L^\infty(\Omega)$ and either $\rho(x) > \lambda_{1,p}$ for a.e. $x \in \Omega$ or $\rho(x) < \lambda_{1,p}$ for a.e. $x \in \Omega$, then the problem*

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \rho|u|^{p-2}u \text{ on } \Omega \tag{21}$$

does not admit a nonzero solution $u \in W_0^{1,p}(\Omega)$ such that $u \geq 0$.

Proof: If $\rho < \lambda_{1,p}$ a.e. on Ω and $u \in W_0^{1,p}(\Omega)$ is a nonzero weak solution of (21), then

$$\int_\Omega |\nabla u|^p \,\mathrm{d}x = \int_\Omega \rho|u|^{p-2}u \,\mathrm{d}x < \lambda_{1,p}\int_\Omega |u|^{p-2}u \,\mathrm{d}x,$$

which gives $\lambda_{1,p} > \int_\Omega |\nabla u|^p \,\mathrm{d}x / \int_\Omega |u|^p \,\mathrm{d}x$, a contradiction with the Rayleigh formula.

In the case $\rho > \lambda_{1,p}$ a.e. on Ω, we observe that if u is a weak solution of (21), then u is a weak solution of

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda_{1,p}|u|^{p-2}u + h \quad \text{on} \quad \Omega$$

with $h := (\rho - \lambda_{1,p})|u|^{p-2}u$. Clearly, $h \geq 0$ and $h \in L^\infty(\Omega)$, since $u \in L^\infty(\Omega)$ due to Lemma 4.7. Hence, Lemma 4.8 leads to a contradiction ending the proof. □

Proof of Theorem 4.4: Assume that $\rho > \lambda_{1,p}$ a.e. on Ω and fix $\tilde\lambda > \lambda_{1,p}$. Define $H : X_p \times [0,1] \to X_p$ by $H(u,t) := (t\tilde\lambda + (1-t)\rho)N_p(u)$, $u \in X_p$, $t \in [0,1]$. In view of Lemma 4.9, $-A_p(u) + H(u,t) \neq 0$ for all $u \in D(A_p) \setminus \{0\}$ and $t \in [0,1]$. Therefore, we can use the homotopy invariance – Theorem 2.5 (iii) to get

$$\operatorname{Deg}_{M_p}(A_p, \rho N_p, B_{M_p}(0,R)) = \operatorname{Deg}_{M_p}(A_p, \tilde\lambda N_p, B_{M_p}(0,R)). \tag{22}$$

In a similar manner one can prove the same formula in the case $\rho < \lambda_{1,p}$ a.e. on Ω with $\tilde\lambda < \lambda_{1,p}$.

Now we shall prove that conditions $(\mathcal{M}_1)$ and $(\mathcal{M}_2)$ of Theorem 2.6 are satisfied. Observe that, in view of Lemma 4.9, for any $\lambda \neq \lambda_{1,p}$, the eigenvalue problem (19) has no nontrivial and nonnegative weak solutions, i.e. $(\mathcal{M}_1)$ holds. To show $(\mathcal{M}_2)$ let $\tau_0 \in L^p(\Omega)$ be the functional determined by $|u_0|^{p-2}u_0$ with u_0 being a fixed positive solution of the eigenvalue problem (19) with $\lambda = \lambda_{1,p}$. Suppose that there exists $u \in (A_p - \lambda N_p)^{-1}(\{\tau_0\}) \cap M_p$ for some $\lambda > \lambda_{1,p}$. This means that $u \in W_0^{1,p}(\Omega)$ is a nonnegative weak solution of

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda_{1,p}|u|^{p-2}u + h \quad \text{on} \quad \Omega$$

with $h := (\lambda - \lambda_{1,p})|u|^{p-2}u + |u_0|^{p-2}u_0$. It follows from Lemma 4.7 that $h \in L^\infty(\Omega)$. Since $h \geq 0$, Lemma 4.8 implies that such a solution does not exist, a contradiction proving $(\mathcal{M}_2)$. Hence, by Theorem 2.6 and (22), the desired formula follows. □

The obtained formula results in the following general one.

Theorem 4.10 *Let f and F be as in Proposition 4.2 and suppose that (2) hold.*

(i) *If ρ_0 is as in (3) and either $\rho_0(x) < \lambda_{1,p}$, for a.e. $x \in \Omega$, or $\lambda_{1,p} < \rho_0(x)$, for a.e. $x \in \Omega$, then there exists $\delta > 0$ such that $A_p(u) \neq F(u)$ for all $u \in D(A_p) \cap \big(B_{M_p}(0,\delta) \setminus \{0\}\big)$ and*

$$\operatorname{Deg}_M(A, F, B_M(0,\delta)) = \begin{cases} 1, & \text{if} \quad \rho_0(x) < \lambda_{1,p} \quad \text{for a.e. } x \in \Omega, \\ 0, & \text{if} \quad \rho_0(x) > \lambda_{1,p} \quad \text{for a.e. } x \in \Omega. \end{cases}$$

(ii) *If ρ_∞ is as in* (3) *either $\rho_\infty(x) < \lambda_{1,p}$, for a.e. $x \in \Omega$, or $\lambda_{1,p} < \rho_\infty(x)$, for a.e. $x \in \Omega$, then there exists $R > 0$ such that $A_p(u) \neq F(u)$ for all $u \in D(A_p) \cap \big(M_p \setminus B_{M_p}(0,R)\big)$ and*

$$\mathrm{Deg}_{M_p}(A_p, F, B_{M_p}(0,R)) = \begin{cases} 1, & \textit{if} \quad \rho_\infty(x) < \lambda_{1,p} \quad \textit{for a.e. } x \in \Omega, \\ 0, & \textit{if} \quad \rho_\infty(x) > \lambda_{1,p} \quad \textit{for a.e. } x \in \Omega. \end{cases}$$

Proof: (i) Define $H : M_p \times [0,1] \to X_p$ by $H(u,t) := tF(u) + (1-t)\rho_0 N_p(u)$, $(u,t) \in M_p \times [0,1]$. By Proposition 4.2, H is continuous and $F \circ N_p^{-1}$ is tangent to M^*. Moreover we claim that

(23) there is $\delta > 0$ such that $-A_p(u) + H(u,t) \neq 0$ for all $u \in M_p \cap D(A_p)$, $t \in [0,1]$.

Suppose to the contrary that there exists (u_n) in $(M_p \cap D(A_p)) \setminus \{0\}$ and (t_n) in $[0,1]$ such that $u_n \to 0$ in X_p and $-A_p(u_n) + H(u_n, t_n) = 0$, $n \geq 1$. Then clearly, if we put $w_n := \|u_n\|_{X_p}^{-1} u_n$ and $s_n := \|u_n\|_{X_p}$, then $A_p(w_n) = s_n^{1-p} H(s_n w_n, t_n)$, which gives

$$w_n = J_1\left(N_p(w_n) + s_n^{1-p} H(s_n w_n, t_n)\right), \ n \geq 1. \tag{24}$$

The growth condition (2) and the existence of the first limit in (3) imply that there exists $C_1 > 0$ such that $\|N_p(w_n) + s_n^{1-p} H(s_n w_n, t_n)\|_{X_p^*} \leq C_1$ for all $n \geq 1$. Therefore we infer that (w_n) has a subsequence convergent in X_p, since, according to Proposition 4.1 and Proposition 3.1 (v), J_1 is completely continuous. In the sequel, we may assume that (w_n) converges almost everywhere to some $w_0 \in M_p \setminus \{0\}$ and that one has $g \in X_p$ such that $|w_n| \leq g$ a.e. on Ω. Further, note that if $w_n(x) \neq 0$, then

$$\frac{f(x, s_n w_n(x))}{s_n^{p-1}} = \frac{f(x, s_n w_n(x))}{(s_n w_n(x))^{p-1}} (w_n(x))^{p-1} \to \rho_0(x)(w_0(x))^{p-1} \text{ as } n \to +\infty,$$

which, by the dominated convergence theorem, implies that $s_n^{1-p} H(s_n w_n, t_n) \to \rho_0 N_p(w_0)$ in X_p^*. Hence, a passage to the limit in (24) yields $w_0 = J_1(N_p(w_0) + \rho_0 N_p(w_0))$, i.e. $-A_p w_0 + \rho_0 N_p(w_0) = 0$. This is a contradiction due to Lemma 4.9 and (23) is proved.

Clearly, (24) allows us to use the homotopy invariance – Theorem 2.5 (iii) to see that $\mathrm{Deg}_{M_p}(A_p, F, B_{M_p}(0,R)) = \mathrm{Deg}_{M_p}(A_p, \rho_0 N_p, B_{M_p}(0,R))$, which together with Theorem 4.4 provides the required formula.

(ii) The proof is analogical to that for part (i) and it is left to the reader. $\square$

Proof of Theorem 1.1: Let $\delta > 0$ and $R > \delta$ be like in Theorem 4.10. Then by use of the additivity property – Theorem 2.5 (ii), we get

$$\begin{aligned} \mathrm{Deg}_{M_p}(A_p, F, B_{M_p}(0,R) \setminus \overline{B_{M_p}(0,\delta)}) &= \mathrm{Deg}_{M_p}(A_p, F, B_{M_p}(0,R)) - \mathrm{Deg}_{M_p}(A_p, F, B_{M_p}(0,\delta)) \\ &= \begin{cases} 1, & \text{if} \ \ \rho_0(x) > \lambda_{1,p} > \rho_\infty(x) \ \ \text{for a.e. } x \in \Omega, \\ -1, & \text{if} \ \ \rho_0(x) < \lambda_{1,p} < \rho_\infty(x) \ \ \text{for a.e. } x \in \Omega. \end{cases} \end{aligned}$$

Hence the existence property of the topological degree gives the existence of $u \in B_{M_p}(0,R) \setminus \overline{B_{M_p}(0,\delta)}$ such that $A_p(u) = F(u)$, which is a required nonzero nonnegative weak solution of (1). $\square$

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